TASI Lecture 3: Generalized Abelian Gauge Theory

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1. Overview Of Lectures 3 + 4

2. Classical Generalized Maxwell Theory

Begin with 4d Maxwell on $\mathbb{M}^{1,3}$
 $\quad\quad\quad\quad\quad$ $=$ 3+1 Mink. space

Fieldstrength $F \in \Omega^2(\mathbb{M}^{1,3})$

Spacetime splitting $\Longrightarrow$ electric/magnetic field decomposition

$$F = F_m + F_e = \frac{1}{2}\epsilon_{ijk} B_i dx^j \wedge dx^k + E_i dx^i dx^0$$

Vacuum Maxwell equations

$$dF = 0 \underset{\text{exercise!}}{\Longleftrightarrow} \begin{cases} \vec{\nabla}\cdot\vec{B} = 0 \\ \dfrac{\partial B}{\partial x^0} + \vec{\nabla}\times\vec{E} = 0 \end{cases}$$

To write the other two Maxwell equations we need the Hodge $*$ operator:

Hodge $*$: Let $(M_n, g_{\mu\nu})$ be an n-dimensional manifold with a nondegenerate metric $g_{\mu\nu} \in \Gamma(\mathrm{Sym}^2 T^* M_n)$ of any signature.

Assume M_n is orientable and choose an orientation: $\Longrightarrow$

$\exists$ $\mathrm{vol}(g) \in \Omega^n(M_n)$, nowhere zero

In local coordinates:

$$\mathrm{vol}(g) = \sqrt{|\det g_{\mu\nu}|}\, \underbrace{dx^1 \wedge \cdots \wedge dx^n}_{\text{determined by orientation of } T_p^* M_n}$$

Now we define a linear operator

$$* : \Omega^\ell(M_n) \longrightarrow \Omega^{n-\ell}(M_n)$$

To do this we first introduce the local inner product on $\Lambda^\ell T_p^* M_n$

For:

$$\alpha = \frac{1}{l!} \alpha_{\mu_1 \dots \mu_l} dx^{\mu_1} \wedge \text{———} \wedge dx^{\mu_l} \in \Lambda^l T_p^* M_n$$

$$(\alpha, \beta)_p := \frac{1}{l!} g^{\mu_1\nu_1} \cdots g^{\mu_l\nu_l} \alpha_{\mu_1 \dots \mu_l}^{(p)} \beta_{\nu_1 \dots \nu_l}^{(p)}$$

Then the formula:

$$\boxed{\alpha \wedge * \beta := (\alpha, \beta) \operatorname{vol}(g)}$$

<u>defines</u> $*\beta$ since it holds for all α.

<u>Note</u>: $\alpha \wedge * \beta = \beta \wedge * \alpha$

<u>Exercises</u>: 1 $*^2 : \Omega^l \longrightarrow \Omega^l$

acts as multiplication by the sign

$$*^2\big|_{\Omega^l} = (-1)^{l(n-l)} \cdot \operatorname{sign}(\det g_{\mu\nu})$$

(2) In local coordinates

$$* \left(dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k} \right) = \frac{|\det g_{\mu\nu}|^{1/2}}{(n-k)!} \epsilon^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_{n-k}} \, dx^{\nu_1 \cdots \nu_{n-k}}$$

with $\epsilon_{1 \cdots n} = +1$.

(3) For a product metric $g_1 \oplus g_2$ on $M_1 \times M_2$ and product orientation $\mathrm{vol}(g_1) \wedge \mathrm{vol}(g_2)$, $\omega_1 \in \Omega^{k_1}(M_1)$ and $\omega_2 \in \Omega^{k_2}(M_2)$

$$*(\omega_1 \wedge \omega_2) = (-1)^{k_2(n_1 - k_1)} (*_{g_1} \omega_1) \wedge (*_{g_2} \omega_2)$$

$n_1 = \dim M_1$

(4) Under conformal transformation

$$g'_{\mu\nu} = \lambda^2 g_{\mu\nu} \Rightarrow *_{g'_{\mu\nu}} \Big|_{\Omega^k} = \lambda^{n-2k} *_{g_{\mu\nu}}$$

(5) $\mathbb{M}^{1,1}$ $g_{\mu\nu} dx^\mu dx^\nu = - dx^0 \otimes dx^0 + dx^1 \otimes dx^1$

Orientation: $\mathrm{vol}(g) = dx^1 \wedge dx^0$

$$* dx^\pm = \pm dx^\pm , \qquad x^\pm := x^0 \pm x^1$$

(6) $\mathbb{M}^{1,3}$ $g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}$ $\mathrm{vol}(g) = dx^{0123}$

$$* (dx^0 \wedge dx^i) = -\frac{1}{2} \epsilon^{ijk} dx^j \wedge dx^k$$

$$* (dx^i \wedge dx^j) = \epsilon_{ijk} dx^0 \wedge dx^k$$

(7) On Euclidean $\mathbb{R}^D$ with orientation:

$$dx^1 \wedge \cdots \wedge dx^D = r^{D-1} dr \wedge \Omega_{D-1}$$

$$\Rightarrow \quad * dr = r^{D-1} \Omega_{D-1}$$

$$\Rightarrow \quad * d\left(\frac{1}{r^{D-2}}\right) = -(D-2)\Omega_{D-1}$$

Introduce unit volume form on

S^{D-1}: $\omega_{D-1} = \Omega_{D-1} / v_D$

$$v_D = 2\pi^{D/2}/\Gamma(D/2)$$

$$d\omega_{D-1} = \eta\left(\{0\} \hookrightarrow \mathbb{R}^D\right)$$
$$= \delta^D(0)\, dx^1 \wedge \cdots \wedge dx^D$$

$$\Rightarrow \quad d\left(\frac{1}{v_D} * \frac{dr}{r^{D-1}}\right) = \eta\left(\{0\} \hookrightarrow \mathbb{R}^D\right)$$

Solution to exercise 1:

Try to solve it yourself first!

Choose an ON basis for $T_p^* M_n$

$e^1, \cdots e^n$ so that the orientation volume form is $e^1 \wedge \text{—} \wedge e^n$ and

$$(e^\alpha, e^\beta) = \eta^\alpha \delta^{\alpha\beta} \qquad \eta^\alpha \in \{\pm 1\}$$

For a multi-index $I = (\alpha_1 < \alpha_2 < \cdots < \alpha_\ell)$

Let $I_c = (\beta_1 < \beta_2 < \cdots < \beta_{n-\ell})$ be the complementary multi-index.

Define a sign $s(I, I_c)$ by

$$e^I \wedge e^{I_c} = s(I, I_c)\, e^1 \wedge \text{———} e^n$$

Note that $e^I \wedge e^{I_c} = (-1)^{\ell(n-\ell)} e^{I_c} \wedge e^I$

so

$$s(I, I_c)\, s(I_c, I) = (-1)^{\ell(n-\ell)}$$

Now
$$*e^{I} = \eta^{\alpha_1} \cdots \eta^{\alpha_\ell} s(I, I_c)\, e^{I_c}$$
$$*e^{I_c} = \eta^{\beta_1} \cdots \eta^{\beta_{n-\ell}} s(I_c, I)\, e^{I}$$

so
$$*^2 e^{I} = \eta^{1} \cdots \eta^{n} (-1)^{\ell(n-\ell)} e^{I}$$
$$= \operatorname{sgn}(\det g_{\mu\nu}) (-1)^{\ell(n-\ell)} e^{I}$$

$\blacksquare$

$2^{\underline{nd}}$ Maxwell equation (in vacuum):

$$d * F \underset{\text{exercise!}}{\iff} \begin{cases} \vec{\nabla}\cdot\vec{E} = 0 \\ \dfrac{\partial \vec{E}}{\partial x^0} - \vec{\nabla}\times\vec{B} = 0 \end{cases}$$

Now generalize to arb. manifolds with nondegenerate metric: (Euc. or Lor. signature)

$(M_n, g_{\mu\nu})$: $\quad F \in \Omega^{\ell}(M_n)$ "generalized Maxwell theory"

E.O.M $\quad dF = 0 \;\;\&\;\; d(*F) = 0$

N.B. Electromagnetic (Hodge) duality:

$\tilde{F} = *F \in \Omega^{n-\ell}(M_n)$ satisfy a pair of equations of the same type with $\ell \leftrightarrow (n-\ell)$

<u>Ex</u>: For $M^{1,3}$ work out $\tilde{F}$ in terms of $\vec{E}$ and $\vec{B}$

Solutions:

Constant tensor

(A.) Lorentz $M^{1,n-1}$ $F = \wp\, e^{ik\cdot x}$

Show $k^2 = 0$: speed of light

Solution space is an ∞-dimensional linear space.

(B.) Eucl. signature: Riemannian (M_n, g) & M_n compact.

Hodge decomposition: Put a nondegenerate inner product on $\Omega^*(M_n)$

$$(\alpha, \beta) := \int_{M_n} \alpha \wedge * \beta$$

Then $d^{\dagger} = \pm * d *$

We have an orthogonal decomposition

$$\Omega^k(M_n) = \mathcal{H}^k(M_n) \oplus \mathrm{Im}\left(d : \Omega^{k-1} \to \Omega^k\right)$$
$$\oplus\ \mathrm{Im}\left(d^\dagger : \Omega^{k+1} \longrightarrow \Omega^k\right)$$

$\mathcal{H}^k(M_n) =$ vector space of <u>harmonic</u> <u>forms</u> $d\alpha = 0$ & $d^\dagger\alpha = 0$

An important thm we'll use a lot is the Hodge theorem

$$\mathcal{H}^k(M_n) \cong H^k_{dR}(M_n)$$

$$:= \frac{\ker\left(d : \Omega^k \longrightarrow \Omega^{k+1}\right)}{\mathrm{im}\left(d : \Omega^{k-1} \longrightarrow \Omega^k\right)}$$

Fin-diml real vector space.

Remark 1: There is a nice connection to Ken Intriligators lectures. If M_n is the target space of a SQM then the Hilbert space of the SQM is $\Omega^*_{L^2}(M_n)$, The susy operators are $d, d^\dagger$ and the supersymmetric groundstates are the harmonic forms.

Here we are studying dynamical fields on a spacetime M_n.

Same mathematics. Very different physics.

Remark 2: An Important Generalization Of Generalized Maxwell Theory: The Self-Dual Field.

Suppose $n = 2\ell$

$$* \, \Omega^\ell \to \Omega^\ell \qquad *^2 = (-1)^\ell \, \mathrm{sgn}(\det g_{\mu\nu})$$

$\Rightarrow$ We can impose self-duality eqs on real fieldstrengths:

$F = *F$ (SD field) <u>OR</u> $F = -*F$ (ASD field)

Consistency $\Rightarrow$ Euclidean sign: $n = 0 \bmod 4$
Lorentz sign: $n = 2 \bmod 4$

If we also impose $dF = 0$ then the other Maxwell equation comes for free. This is the classical theory of the (anti-) self dual field

Example: $\mathbb{M}^{1,1}$, $-(dx^0)^2+(dx^1)^2$; $dx^1\wedge dx^0$

$$F = \pm *F \implies F = \underbrace{\phi(x^0;x^1)}_{\mathbb{R}\text{-valued function}}\, dx^{\pm}$$

$$dF = 0 \implies \partial_{\mp}\phi = 0$$

massless chiral scalar.

Action Principle: Let us return to the nonselfdual field. To write an action we must break electromagnetic duality. We want to use Stokes' lemma so if we prefer $F \in \Omega^{\ell}$ to $\tilde{F} \in \Omega^{n-\ell}$ then

$dF = 0 \Rightarrow$ locally we can write $F = dA$

$A \in \Omega^{\ell-1}(\mathcal{U})$

local patch

Obstruction to writing $F = dA$ globally is

$$H^{\ell}_{dR}(M) = \ker\left(d : \Omega^{\ell} \to \Omega^{\ell+1}\right) / \operatorname{im}\left(d : \Omega^{\ell-1} \to \Omega^{\ell}\right)$$

For a fixed coh class h choose a representative F_0 with $h = [F_0]$

Write: $F = F_0 + dA \qquad A \in \Omega^{\ell-1}(M_n)$

For all other closed F in that class. Then on that set we have

$$S_h[A] = \int \lambda \; F * F$$

Stationarity $\Rightarrow$ $d(*F) = 0$

Hodge Theorem: For Eucl. signature $\exists !$ minimum of the action.

Action also gives coupling to gravity and hence energy-momentum:

$$T_F \in \Gamma(\mathrm{Sym}^2 T^* M)$$

Exercise (a) If $v \in TM$ show that

$$\lambda^{-1} T_F(v)\Big|_p = (\iota_v F, \iota_v F)_p - \tfrac{1}{2}(v,v)_p (F,F)_p$$

(b.) $T_F = T_{\tilde{F}}$

(generalizes duality invariance of $E^2 + B^2$ & $E \times B$)

Remarks:

1. In the self-dual case $F = \pm * F$ and $n = 2 \bmod 4$ we have $\ell = 1 \bmod 2$ is <u>odd</u> hence $\int F \wedge * F = \int F \wedge F = 0$

Hence there is no obvious Lorentz-invt action. There do exist (many) actions for the self-dual field, but much more needs to be said. It is important here that we are working with Abelian gauge theory.

Standard folk-wisdom states that there is no description of a <u>nonabelian</u> analog in terms of elementary fields and a local action. (There are local fields in the nonabelian analog, e.g. the energy-momentum tensor.)

② Kaluza-Klein reduction shows it is much more natural to consider collections of generalized Maxwell fields with action

$$\exp \frac{i}{\hbar}\left[\int \lambda_{ij} F^{i} * F^{j} + \theta_{ij} F^{i} \wedge F^{j}\right]$$

nondeg. symmetric bilinear form

symmetric or antisymmetric

So it is natural to generalize generalized Maxwell theory to $F \in \bigoplus_{p,q} \Omega^{p}(M_n, V^{q})$ for a $\mathbb{Z}$-graded vector space $V^{\cdot}$ equipped with suitable bilinear forms.

(3) In the case $\ell = 1$ we can write $F = d\phi$ locally but the scalar field might not be globally defined.

If we take $\phi \in \mathbb{R}/2\pi\mathbb{Z}$ then the factor λ has the meaning of radius-squared of the target space circle of ϕ.

(In general the nonlinear σ-model action

$$\int g_{ij}(X)\, h^{\mu\nu}\, \partial_\mu X^i\, \partial_\nu X^j\ \mathrm{vol}(h)$$

shows that the kinetic term defines a length^2 on target space)

3. Electric & Magnetic Currents

<u>Vacuum</u> $dF = 0$ & $d * F = 0$

Currents $J_m \in \Omega^{\ell+1}$ $J_e \in \Omega^{(n-\ell)+1}$

Response of field to backgnd current:

$$dF = J_m \qquad d(*F) = J_e$$

<u>Note</u>:

1. $J_m \neq 0 \Rightarrow F \neq dA$. Magnetic current obstructs the existence of a gauge potential.

2. $dJ_m = dJ_e = 0$ Current conservation

3. Smooth $F \Rightarrow J_m, J_e$ cohomologically trivial.

$\Rightarrow$ Puzzle: Shouldn't the cohomology class of J_e somehow measure charge?

Introduce notion of relative cohomology: (See Bott+Tu): Given an inclusion $i: A \hookrightarrow X$

The relative chain complex is

$$C^k_{dR}(X,A) = \Omega^k(X) \oplus \Omega^{k-1}(A)$$

differential $d(\omega,\theta) = (d\omega,\ i^*\omega - d\theta)$

Exercise: (a) Check $d^2=0$

(b.) Show that "closed" means $d\omega=0$ and $\omega|_A$ is trivialized.

<u>Define</u> $H^k_{dR}(X,A) := \ker d / \operatorname{im} d$

Now

$$0 \to \Omega^{k-1}(A) \to C^k(X,A) \to \Omega^k(X) \to 0$$

$$\theta \longmapsto (0,\theta)$$

$$(\omega,\theta) \longmapsto \omega$$

induces LES in cohomology:

$$\xrightarrow{} H^{k-1}(A) \longrightarrow H^k(X,A) \longrightarrow H^k(X) \xrightarrow{\text{restriction}} H^k(A) \longrightarrow \cdots$$

Note that $(J_e, *F)$ is exact, but $[(J_e, 0)]$ could be a nontrivial class in

$$H^{n-l+1}_{dR}(M, M^{-J_e})$$

$$M^{-J_e} := M - \mathrm{Supp}(J_e)$$

$$\cdots \longrightarrow H^{n-l}(M) \xrightarrow{z^*} H^{n-l}(M^{-J_e}) \xrightarrow{\delta} H^{n-l+1}(M, M^{-J_e})$$

$$\xrightarrow{\psi} H^{n-l+1}(M) \longrightarrow \cdots$$

$$\ker\psi \simeq \operatorname{im}\delta \cong H^{n-l}(M^{-J_e}) \big/ z^* H^{n-l}(M)$$

The electric charge group is, by definition the kernel of ψ and by the LES

$$Q_e \cong H^{n-\ell}_{dR}(M^{-J_e}) / \imath^* H^{n-\ell}(M_e)$$

$H^{n-\ell}_{dR}(M_e)$: Classical flux group — these are fluxes not sourced by charge.

This is the mathematical expression of the idea that the charge is measured by the "flux at ∞".

If $M_n = \mathbb{R}_t \times N_{n-1}$ and the support of $\overline{U}_e$ is compact for all time we can identify Q_e as the kernel of

$$H^{n-\ell+1}_{cpt}(N_{n-1}) \longrightarrow H^{n-\ell+1}(N_{n-1})$$

4. Branes

p-branes are extended objects generalizing point particles with worldvolumes:

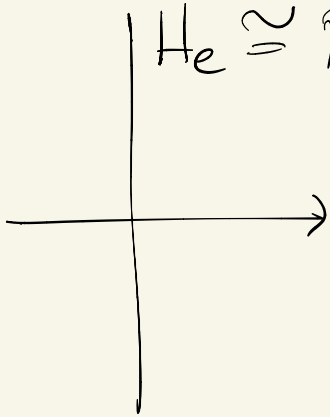
$$W_e = S_p \times \mathbb{R}_t \subset N_{n-1} \times \mathbb{R}_t = M_n$$

In a generalized Maxwell theory electrical branes can be viewed as objects which produce an electric current:

$$J_e = q_e \underbrace{\eta(W_e \hookrightarrow M_n)}_{\substack{n-(p_e+1) \text{ form} \\ \text{Poincaré dual to } W_e}}$$

$$d * F = J_e \implies p_e = l - 2$$

Solution when $S_p = H_e$ is a hyperplane in spatial $\mathbb{R}^{n-1}$



$H_e \cong \mathbb{R}^p$

$H_e^{\perp} \cong \mathbb{R}^{D_e}$

$$D_e = (n-1) - p = n - \ell + 1$$

$$F = \frac{q_e}{V_e} \, \frac{dt \wedge dr}{r^{D_e - 1}} \wedge \mathrm{vol}(H_e)$$

$$\begin{array}{ccc}
H^0(W) & \xrightarrow{2*} & H^{n-\ell+1}(M_n, M_n - W) \\
\rotatebox{90}{$\cong$} & & \rotatebox{90}{$\in$} \\
\mathbb{R} \ni q_e & \xmapsto{2*} & [(J_e, 0)]
\end{array}$$

Similarly magnetic branes would have a world volume

$$W_m = \underset{p_m\text{-dim}}{S_m} \times \mathbb{R} \subset N_{n-1} \times \mathbb{R}$$

and produce a magnetic current:

$$dF = g_m \underbrace{\eta(W_m \hookrightarrow M_n)}_{(l+1)\text{-form}} \Rightarrow p_m = n - l - 2$$

In $M^{1,n}$

$$F = 2\pi g_m \, \omega_\perp$$

$\omega_\perp$ unit volume of $S^l \subset H_m^\perp \cong \mathbb{R}^{l+1}$

5. Action For Test Branes

Key insight of Joe Polchinski:

Branes are not just soliton solutions in supergravity but are dynamical objects which must be included in the description of the space of states in string theory.

Unlike <u>defects</u>, branes can wiggle and move. These are among their internal degrees of freedom.

So — similar to the AdS/CFT correspondence — we can change our point of view and consider the world volume theory of a test brane moving in a prescribed field configuration F.

Let's start with a point electric charge in $d=2$ Maxwell on a Lorentz signature spacetime M_n with prescribed field F

For an unchanged particle the path integral has "DBI" action

$$\exp \frac{i}{\hbar} \int_{W_1} T\, ds \qquad ds = \sqrt{-\left(\frac{dx}{d\tau}\right)^2}\, d\tau$$

τ = time coordinate along world line

T = "tension" = mass of particle

EOM = geodesic eq:

$$\frac{d}{d\tau}\left(T \frac{dx_\mu}{ds}\right) = 0$$

Lorentz force: If particle has charge q_e:

$$(*) \quad \frac{d}{d\tau}\left(T\frac{dx_\mu}{ds}\right) = q_e F_{\mu\nu}\frac{dx^\nu}{d\tau}$$

$\Longrightarrow$ new action

$$\exp\left(\frac{i}{\hbar}\int_{W_1} T\,ds\right)\chi(W_1)$$

If $F = dA$ $\quad A \in \Omega^1(M_n)$

$$\chi(W_1) = \exp\left(\frac{i}{\hbar}\int_{W_1} q_e A\right)$$

produces eq $(*)$ above

Now we note two key properties which will be defining properties of the map $\chi : \begin{Bmatrix} \text{world} \\ \text{lines} \end{Bmatrix} \to U(1)$

(A) If we have multiple particles then the actions should add so if we replace the worldline by $W_1 \amalg W_1'$ then

$$\chi(W_1 \amalg W_1') = \chi(W_1)\chi(W_1')$$

So if we restrict to 1-cycles

$$\chi \in \mathrm{Hom}\left(Z_1(M), U(1)\right)$$

is a homomorphism of Abelian groups.

(B) If we have a (closed) worldline and it is of the form: $W_1 = \partial B_2$ for some disc then

$$\chi(W_1) = \exp\left(\frac{i}{\hbar}\int_{B_2} q_e F\right)$$

These physical considerations motivate the mathematical definition

Def. A Cheeger-Simons character of degree 2 is a group homom:

$$\chi \in \mathrm{Hom}\left(Z_1(M_n), U(1)\right)$$

with the special property that there exists $F \in \Omega^2(M_n)$ such that, whenever $W_1 \in Z_1(M_n)$

is the boundary of a 2-chain:

$$\mathcal{W}_1 = \partial \mathcal{B}_2$$

$$\chi(W_1) = \exp\left(i \int_{\mathcal{B}_2} F\right)$$

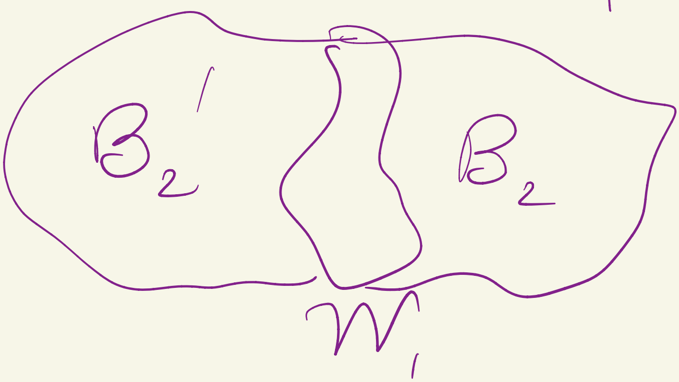
The set of such characters has a natural Abelian group structure

$$(\chi_1 \cdot \chi_2)(W) := \chi_1(W)\,\chi_2(W)$$

and forms an ∞-dim'l Abelian group denoted $\check{H}^2(M)$ known as the Cheeger-Simons or differential cohomology group of degree $l=2$.

Note: (1) We have absorbed the $q_e/\hbar$ into F. So "F" in our physical motivation is not exactly the same as "F" in the definition of the Cheeger-Simons group.

(2) (A)+(B) $\Longrightarrow$ quantization of periods of F:



$\Sigma_2 = B_2 \cup B_2'$ is a closed 2-cycle in M_n $\Longrightarrow$ the fieldstrength F of part (B) satisfies $\exp\left(i \int_{\Sigma_2} F\right) = 1$

$$\Longrightarrow \quad F \in \Omega^2_{\mathbb{Z}'}(M_n)$$

Remarks:

1. Any 2-form which has all periods of 2-cycles in $2\pi\mathbb{Z}$ must be closed:

$$\Omega^2_{\mathbb{Z}'}(M_n) \hookrightarrow \Omega^2_d(M_n) := \ker(d)$$

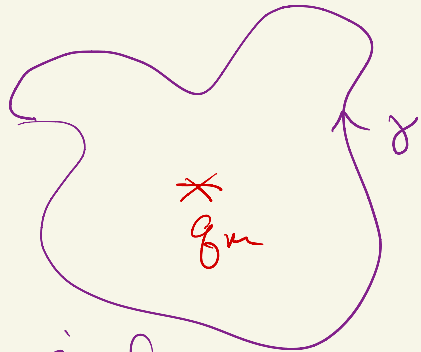
2. The argument above is closely related to Dirac's quantization argument for product of electric + magnetic charge

$$\frac{g^2}{\hbar c}\frac{e^2}{\hbar c} = \left(\frac{s}{2}\right)^2$$

$$s \in \mathbb{Z}$$

Putting back the $\hbar$ and q_e

The world line action on $\mathbb{R}^3-\{0\}$ has action



$$e^{\frac{i}{\hbar}\int_\gamma q_e A}$$

$$= e^{\frac{i}{\hbar}\int_{D_+} q_e F} = e^{-\frac{i}{\hbar}\int_{D_-} q_e F}$$

but $D_+ \cup D_-$ = closed 2-cycle enclosing magnetic source $F = q_m \omega_2$

$$e^{\frac{i}{\hbar}\int_{D_+\cup D_-} q_e q_m \omega_2} \Rightarrow \frac{q_e q_m}{\hbar} \in 2\pi\mathbb{Z}$$

The q_e, q_m are proportional to charges e, g that appear in Coulomb's law by factors, depending on $2, \pi$.

Nontrivial Fact: Every CS character is the holonomy function of some connection on some principal $U(1)$ bundle over M.

$$\chi(W) = \mathrm{Hol}(\nabla, W)$$

for some connection ∇ on some principal $U(1)$ bundle $P \xrightarrow{U(1)} M$. Informally we can write $\nabla = d + A$ and $\mathrm{Hol}(\nabla, W) = \exp\left(i \int_W A\right)$ but A is not globally well-defined.

The holonomy is gauge invariant and in fact the holonomy function on $Z_1(M)$ carries all the gauge inv't information:

By the holonomy function we mean:

$$\mathrm{Hol}_\nabla : Z_1(M) \longrightarrow U(1)$$

$$\nabla = d + A \,, \qquad \begin{array}{c} \cup \\ W_1 \end{array} \longmapsto \mathrm{Hol}(\nabla, W_1)$$

$$= \text{“}\exp i \int_W A\text{”}$$

This will follow from properties of the group $\check{H}^2(M)$ derived below together with the following very useful theorem about connections on principal G-bundles for compact gauge group G:

<u>Theorem</u>: Let $P \longrightarrow M$ be a principal G-bundle with connection and define

$$\mathrm{Hol}_\nabla : Z_1(M) \longrightarrow \mathrm{ConjClass}(G)$$

If G is compact and $\mathrm{Hol}_\nabla = \mathrm{Hol}_{\nabla'}$ then $P \cong P'$ and ∇, ∇' are gauge equivalent.

$\boxtimes$ Note: There are counterexamples when G is noncompact. (For $GL(n, \mathbb{C})$ the RH problem gives a counterexample.)

See A. Sengupta, "Gauge Invariant Functions Of Connections," Proc. Am. Math Soc. Vol. 121, pp. 897 – 905

We can now get a picture of the Abelian group $\check{H}^2(M)$.

Note that for each $x \in H^2(M, \mathbb{Z})$ $\exists$ principal $U(1)$ bundle $P_x \to M$ with $c_1(P_x) = x$. Let $(\mathcal{A}/\mathcal{G})_x$ be the set of gauge equivalence classes of connections on P_x. We have

$$\check{H}^2(M) = \coprod_{x \in H^2(M,\mathbb{Z})} (\mathcal{A}/\mathcal{G})_x$$

Now for <u>any</u> principal G bundle for <u>any</u> G

$P \xrightarrow{G} M \qquad \mathcal{A}(P) := \mathrm{Conn}\left(P \xrightarrow{G} M\right)$

is an affine space modeled on $\Omega^1(M; \mathrm{ad}P)$:

Choose a basepoint connection ∇_0. Then every other connection is of the form $\nabla = \nabla_0 + \alpha$

$\alpha \in \Omega^1(M; \mathrm{ad}P)$

For $G = U(1)$ $\mathrm{ad}P = M \times \mathbb{R}$ is trivial

$$\Omega^1(M; \mathrm{ad}P) = \Omega^1(M)$$

$$\nabla = \nabla_0 + \alpha,$$

α globally defined 1-form

Gauge transformations:

$$\alpha \longrightarrow \alpha + \omega \qquad \omega \in \Omega^1_{\mathbb{Z}}(M)$$

"large" ω has nontrivial periods

"small" $\omega = d\epsilon$, $\epsilon \in \Omega^0(M)$

globally well-defined

Now remember Hodge decomposition:

$$\Omega^1 \cong \mathcal{H}^1 \oplus \operatorname{Im} d \oplus \operatorname{Im} d^\dagger$$

$$\Omega^1_{\mathbb{Z}} \cong \mathcal{H}^1_{\mathbb{Z}} \oplus \operatorname{Im} d$$

$$\Omega^1 / \Omega^1_{\mathbb{Z}} \cong \mathcal{H}^1 / \mathcal{H}^1_{\mathbb{Z}} \oplus \operatorname{Im} d^\dagger$$

So — noncanonically — we can write:

$$\check{H}^2(M) \cong T \times \Gamma \times V$$

T = Connected torus $= \mathcal{H}^1/\mathcal{H}^1_{\mathbb{Z}} \cong U(1)^{b_1}$

Γ = fin. generated Abelian group $= H^2(M,\mathbb{Z})$

V = ∞-dim'l vector space:

$$V = \mathrm{Im}\left(d^\dagger : \Omega^2 \longrightarrow \Omega^1\right)$$

Note that if $\alpha \in \mathrm{Im}\, d^\dagger$ Then

$d^\dagger \alpha = 0$ i.e. $\partial^\mu \alpha_\mu$. This is the well-known Landau gauge fixing.